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Using a non-integer moment of the impulse response to estimate the half-space conductivity

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ABSTRACT

The *n*th-order moments of the electromagnetic impulse response are useful for interpreting electromagnetic data. We have derived an analytic expression for the half-order moment of a conductive half-space. By inverting this expression, the measured half-order moment can be used to estimate an apparent conductivity of the ground. The first-order moment can also be used to estimate the half-space conductivity. A sensitivity analysis indicates that for an airborne EM configuration, the half-order moment will be most sensitive to material in the top 26–48 m, while the first-order moment will be sensitive to deeper material (down to depths between 66 and 127 m).

INTRODUCTION

Smith and Lee (2002a) have introduced the concept of the moments of the electromagnetic (EM) impulse response. The nth-order moment M^n is defined as

$$M^n = \int_0^\infty t^n I(t)\,\mathrm{d}t, \tag{1}$$

where t is time and $I(t)$ is the quadrature component of the impulse response. The quadrature component of the impulse response is the impulse response with the delta function at zero time removed. For more discussion on the in-phase and quadrature decomposition of time-domain EM data, see Smith (2001). Smith and Lee (2002a) discuss how to calculate the moments for cases when only an approximation to an impulse response is measured.

In the case of a wire loop, the moments with integer indices have been derived up to an arbitrarily high order (Smith and Lee 2002a). In practice, the sphere in a dipole field can be calculated up to fourth order (Smith and Lee 2002a) and the sphere in a uniform field can be calculated up to 29th order (Smith and Lee 2001). The number of moments for horizontally layered structures is more limited, being restricted to the first- and second-order moments for a thick layer. When the horizontal layer is thin there are analytic expressions for the first- and second-order moments for the z-component and the first-, second- and third-order moments of the x-component (Smith and Lee 2002b). In the case of a half-space, there are analytic solutions only for the first-order moment.

The half-space model is a useful model to use, as it is representative of the geology in a variety of situations; for example, where there is thick uniform conductive overburden or when the surficial layer in a sedimentary basin is largely uniform. A half-space model is an implicit assumption when calculating the apparent conductivity and apparent resistivity, which are quantities used frequently in the interpretation of airborne EM data (Fraser 1978; Palacky and West 1991) and ground resistivity data (Grant and West 1965).

The first-order moments can be used to estimate a half-space apparent conductivity (Smith and Lee 2002b). The interpretation could be improved if there were a second estimate of the conductivity. We introduce the notion of non-integer or fractional moments to help us to achieve this goal. Specifically, we look at the half-order moment, as it is possible to derive an analytic expression for this moment in the case of the half-space model.

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THE HALF-ORDER MOMENT OF A HALF-SPACE

The magnetic-field response of a half-space excited by a vertical dipole source is given by Ward and Hohmann (1988) as

$$H_\rho(\rho, z) = \frac{m_{TX}}{4\pi}\int_0^\infty r_{TE}e^{\lambda(z-h)}\lambda^2 J_1(\lambda\rho)\,d\lambda, \tag{2}$$

$$H_z(\rho, z) = \frac{m_{TX}}{4\pi}\int_0^\infty r_{TE}e^{\lambda(z-h)}\lambda^2 J_0(\lambda\rho)\,d\lambda, \tag{3}$$

where H_j is the magnetic field in the $j = \rho$ or $j = z$ direction, r_{TE} is the reflection coefficient of the TE mode, λ is the Hankel transform variable, m_{TX} is the transmitter dipole moment and J_m is the Bessel function of order m. The symbol h denotes the transmitter height, and when not being used as a subscript, z and ρ are the vertical and horizontal positions of the receiver. The sign convention used is z negative and h positive for receivers above the ground. The term associated with the primary field in equations (4.45) and (4.46) of Ward and Hohmann (1988) has been dropped.

If we assume the quasi-static limit and that the permeability μ is everywhere equal to the free space value, then for a half-space, with conductivity σ and an insulator above, the reflection coefficient, given by Ward and Hohmann (1988), reduces to

$$r_{TE} = \frac{\lambda - u}{\lambda + u}, \tag{4}$$

where $u = \sqrt{\lambda^2 + i\omega\mu\sigma}$, ω is the angular frequency and i is the pure imaginary number.

Both (2) and (3) can be converted to the time domain by setting $p = i\omega$ and taking the inverse Laplace transform:

$$H_j(\rho, z, t) = \frac{m_{TX}}{4\pi} \times \int_0^\infty \left[\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{pt}\frac{\lambda - u}{\lambda + u}dp\right] e^{\lambda(z-h)}\lambda^2 J_m(\lambda\rho)\,d\lambda, \tag{5}$$

where $m = 1$ when $j = \rho$ and $m = 0$ when $j = z$. We denote the inverse Laplace transform integral in square brackets as $I(t)$, so that

$$I(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{pt}\frac{\lambda - u}{\lambda + u}\,dp. \tag{6}$$

We will now consider this integral in more detail. The denominator has a branch point at $p = -\lambda^2/\mu\sigma$. It is possible to deform the contour as shown in Fig. 1 around the branch cut. There are no singularities inside the contour, so the value of the integral along the contour shown must be zero. Because

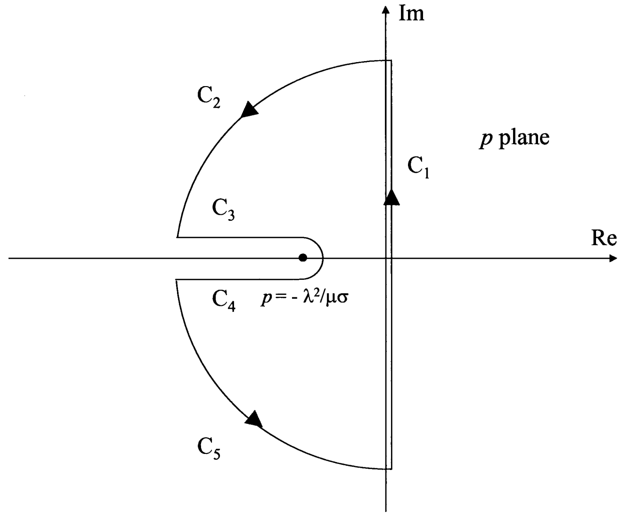

Figure 1 The contour of integration deformed about the branch cut.

the contributions from the quarter circles C_2 and C_5 vanish, the integral up the imaginary axis (C_1) must be the negative of the integral along the two sides of the branch cut, $C_3 + C_4$. By making a change of variable to $s = -\lambda^2/\mu\sigma - p$ it is possible to integrate along both the branches C_3 and C_4 simultaneously, giving

$$I(t) = \frac{-1}{2\pi i}\int_0^\infty e^{-(\lambda^2/\mu\sigma + s)t}\left(\frac{i\sqrt{s\mu\sigma} - \lambda}{i\sqrt{s\mu\sigma} + \lambda} - \frac{-i\sqrt{s\mu\sigma} - \lambda}{-i\sqrt{s\mu\sigma} + \lambda}\right)ds, \tag{7}$$

which can be rearranged to give

$$I(t) = \frac{-1}{\pi}\int_0^\infty e^{-(\lambda^2/\mu\sigma + s)t}\frac{\lambda\sqrt{s\mu\sigma}}{\lambda^2 + s\mu\sigma}\,ds. \tag{8}$$

This is the only time-dependent quantity in the transformed (5). Calculating the half-order moment (carried out below) will require multiplying this quantity by $\sqrt{t}$ and integrating t from zero to infinity, giving

$$\int_0^\infty I(t)\sqrt{t}\,dt = \frac{-2}{\pi}\int_0^\infty \frac{\lambda\sqrt{s\mu\sigma}}{\lambda^2 + s\mu\sigma}\int_0^\infty e^{-(\lambda^2/\mu\sigma + s)t}\sqrt{t}\,dt\,ds. \tag{9}$$

The inner t integral on the right-hand side can be evaluated using equation (6.1.1) in Abramowitz and Stegun (1965):

$$\int_0^\infty e^{-(\lambda^2/\mu\sigma + s)t}\sqrt{t}\,dt = \frac{\sqrt{\pi}}{2(\lambda^2/\mu\sigma + s)^{3/2}}, \tag{10}$$

which allows us to write (9) as

$$\int_0^\infty I(t)\sqrt{t}\,dt = \frac{\lambda(\mu\sigma)^2}{\sqrt{\pi}}\int_0^\infty \frac{\sqrt{s}}{(\lambda^2 + s\mu\sigma)^{5/2}}\,ds. \tag{11}$$

The integral on the right-hand side can be evaluated using the substitution $\sqrt{s} = \lambda/\sqrt{\mu\sigma}\tan\theta$, giving

$$\int_0^\infty I(t)\sqrt{t}\,\mathrm{d}t = \frac{\sqrt{\mu\sigma}}{\lambda\sqrt{\pi}}\int_0^{\pi/2}\sin^2\theta\cos\theta\,\mathrm{d}\theta. \tag{12}$$

Using the integration by parts formula (4.3.127) in Abramowitz and Stegun (1965), we get the simple result,

$$\int_0^\infty I(t)\sqrt{t}\,\mathrm{d}t = \frac{\sqrt{\mu\sigma}}{3\lambda\sqrt{\pi}}. \tag{13}$$

Returning to the moment expression (1), we can now write

$$M_j^{1/2}(\rho, z) = \frac{m_{\mathrm{TX}}}{4\pi}\frac{\sqrt{\mu\sigma}}{3\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-\lambda(h-z)}\lambda \mathrm{J}_m(\lambda\rho)\,\mathrm{d}\lambda. \tag{14}$$

For the $j = z$ case,

$$M_z^{1/2}(\rho, z) = -\frac{m_{\mathrm{TX}}}{4\pi}\frac{\sqrt{\mu\sigma}}{3\sqrt{\pi}}\frac{\partial}{\partial(h-z)}\int_0^\infty \mathrm{e}^{-\lambda(h-z)}\mathrm{J}_0(\lambda\rho)\,\mathrm{d}\lambda. \tag{15}$$

The λ integral has a closed form expression (Wait 1982, p. 117), which yields the desired result,

$$M_z^{1/2}(\rho, z) = \frac{m_{\mathrm{TX}}}{4\pi}\frac{\sqrt{\mu\sigma}}{3\sqrt{\pi}}\frac{h-z}{(\rho^2+(h-z)^2)^{3/2}}. \tag{16}$$

For the $j = \rho$ case,

$$M_\rho^{1/2}(\rho, z) = -\frac{m_{\mathrm{TX}}}{4\pi}\frac{\sqrt{\mu\sigma}}{3\sqrt{\pi}}\frac{\partial}{\partial\rho}\int_0^\infty \mathrm{e}^{-\lambda(h-z)}\mathrm{J}_0(\lambda\rho)\,\mathrm{d}\lambda, \tag{17}$$

which can also be evaluated using the expression in Wait (1982) to give the desired result for this component:

$$M_\rho^{1/2}(\rho, z) = \frac{m_{\mathrm{TX}}}{4\pi}\frac{\sqrt{\mu\sigma}}{3\sqrt{\pi}}\frac{\rho}{(\rho^2+(h-z)^2)^{3/2}}. \tag{18}$$

For the airborne EM problem, the horizontal and vertical locations ρ, h and z are known, as are the other constants. Hence (16) and (18) are in the form,

$$M_j^{1/2}(\rho, z) = G_j(\rho, h, z)\sqrt{\sigma}, \tag{19}$$

which can be inverted very simply, giving

$$\sigma = \left(\frac{M_j^{1/2}(\rho, z)}{G_j(\rho, h, z)}\right)^2. \tag{20}$$

This expression can be used to derive estimates of the conductivity from the half-order moment of either the ρ- or z-component.

Equations (16) and (18) can be compared with the first-order moments for the half-space,

$$M_\rho^1 = \frac{m_{\mathrm{TX}}}{4\pi}\frac{\mu\sigma}{4}\frac{1}{\rho}\left[1 - \frac{h-z}{\sqrt{\rho^2+(h-z)^2}}\right], \tag{21}$$

$$M_z^1 = \frac{m_{\mathrm{TX}}}{4\pi}\frac{\mu\sigma}{4}\frac{1}{\sqrt{\rho^2+(h-z)^2}}. \tag{22}$$

These formulae can also be inverted to estimate the conductivity of the half-space. However, each of the four estimates (two moments, each with two components) will not necessarily give the same answer. Simple inspection of (1) shows that the higher-order moments put more weight on the late-time response and these later times are generally indicative of deeper material (Nabighian 1979). In order to quantify this, we have undertaken a sensitivity analysis. We have used a half-space model and perturbed the conductivity at a range of different depths. The amounts that this perturbation changes half- and first-order moments have been plotted on Fig. 2 (ρ-component) and Fig. 3 (z-component). Each curve has been normalized by the perturbation at the shallowest depth. Note that identical curves are obtained when the conductivity of the half-space is changed. When the normalized sensitivity is large, material at these depths will have a strong influence on the moment and hence the half-space conductivity estimates. We have somewhat arbitrarily taken a normalized sensitivity of 0.3 as a cut-off between material that has a strong influence (>0.3) and material with a weak influence (<0.3). Using this value of the cut-off, the half-order moment of the ρ-component is only influenced by material down to 26 m depth, while the first-order moment is influenced by material down to 66 m. The depth sensitivity for the z-component

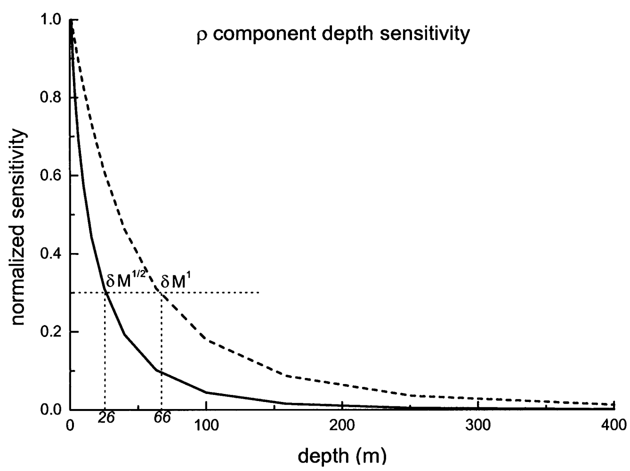

Figure 2 The normalized depth sensitivity of the half- and first-order moments for the ρ-component. The sensitivity is the change in the moment that results when there is a perturbation of the conductivity at a particular depth. The sensitivity at the shallowest depth is used to normalize each curve. For the half-order moment $M^{1/2}$, the sensitivity drops below 0.3 at 26 m depth. The first-order moment M^1 is more sensitive to deeper material, as the sensitivity does not drop below 0.3 until 66 m depth. The nominal horizontal offset of the receiver from the transmitter is $\rho = 130$ m, the vertical offset is $z = -50$ m, and the transmitter height is $h = 120$ m.

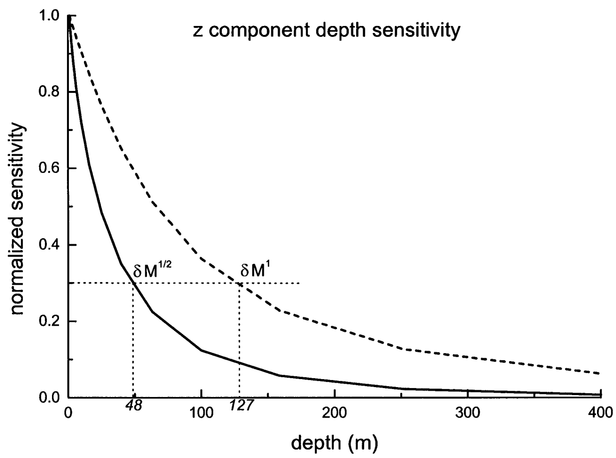

Figure 3 As Fig. 2, showing the sensitivity for the *z*-component. Note that the *z*-component is sensitive to depths about twice as great as the ρ-component.

appears to be roughly twice that of the ρ-component (48 m and 127 m for the half- and first-order moments, respectively). If we had chosen a different cut-off value, the depths would have been slightly different, but the conclusions about the relative depth sensitivity of each moment (and each component) would have been similar.

CONCLUSIONS

For a conductive half-space model it is possible to derive an analytic expression for the half-order moment of the impulse response. The expression is extremely simple and can be easily inverted to estimate the apparent conductivity of the ground from the measured moment.

When the half-space is not homogeneous, the estimated apparent conductivity derived from each moment will be different. The half-order estimate from the ρ-component will reflect the average conductivity in about the top 26 m and the first-order estimate will reflect the material in the top 66 m. The *z*-component sees deeper material (48 and 127 m for the half- and first-order moments, respectively).

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